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Semi-classical periodic-orbit theory for chaotic Hamiltonians with discrete symmetries

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Abstract. We generalize an idea applied recently to the case of identical particles and present a group-theoretical analysis of the periodic-orbit structure of a chaotic dynamical system with a discrete symmetry. The class structure of the group provides the key for the classification of periodic orbits. This structure perfectly fits the quantum-mechanical trace formula which is the starting point for the Balian–Bloch–Gutzwiller semi-classical approximation. For a specific irreducible representation of the symmetry group, we derive a modified form of the periodic-orbit sum.

Dedicated to Klaus Dietrich on the occasion of his sixtieth birthday.

1. Introduction

Discrete symmetry groups are of central importance in quantum mechanics. Their irreducible representations (IR) label the invariant subspaces of Hilbert space which are not connected dynamically. Hence, it is natural to expect that such symmetry groups also play an important role in the phase-space structure of classically chaotic systems, since recent advances in the semi-classical approximation suggest that many (if not all) quantal properties of such systems can be derived semi-classically.

The Bloch–Balian–Gutzwiller approach to semi-classical quantization [1] uses the unstable isolated periodic orbits as the central element of the theory. For a generic fully-chaotic classical system, these orbits form a dense set in phase space and, therefore, determine the dynamical structure. Thus, it is of interest to study the properties of such orbits in the presence of a discrete symmetry, and to use the result for a simplification of Gutzwiller's periodic-orbit sum. This is the programme of the present paper. In our work, we have in mind symmetry groups that relate both to the geometrical properties of the Hamiltonian and to particle identity.

Several authors [2, 3] have approached the problem of group symmetry using the concept of reduced phase space. This concept entails a tiling of the phase space which is chosen in such a way that application of the group operations to a single tile generates the full phase space. This approach appears to be limited to groups of coordinate transformations which are true representations of the full group on the entire phase space, a condition which is usually met. In practical applications, perhaps a more severe difficulty consists in the

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very construction of the tiling. This construction involves the full complexity of the group under consideration. We have in mind, for instance, the case of N identical particles and the associated symmetric group with $N!$ elements. For $N \gg 1$, this group is formidable.

The method developed in the present paper avoids the use of the reduced phase space and generalizes an alternative approach developed earlier by one of the authors [4] for the specific example of permutations of identical particles. We classify the periodic orbits directly in terms of their symmetry properties. This procedure offers the following advantage. In many applications of the periodic-orbit sum, one is only interested in the shortest periodic orbits. Thus, it may never be necessary to take account of the full complexity of the entire group: it will suffice to consider the symmetries that actually matter for the orbits used. For large groups, this may be an essential advantage.

In section 2, we analyse the symmetry properties of the classical periodic orbits in terms of conjugate classes (henceforth, also simply called classes) of the discrete group. We do so without referring to the quantum trace formula for the level density, although it turns out (section 3) that this formula calls for the use of such classes. This procedure leads (section 4) to a restriction of the semi-classical periodic-orbit sum which is commensurate with the minimal invariant subspaces defined by the IR of the group. Finally, we comment on situations where the Hamiltonian contains intrinsic variables such as spin and isospin and draw some conclusions.

2. The structure of the periodic orbits for systems with discrete symmetries

Let H be a Hamiltonian defined on a $2n$-dimensional phase space (symplectic manifold) M, and let H be invariant with respect to a discrete symmetry group $\mathcal{G}$. Let $C_H(t)$ be the Hamiltonian flow induced by H on M, i.e. the canonical transformation which describes the time evolution on M over a time t under the action of H. Clearly, $C_H(t)$ is a map of M onto itself. By assumption, the elements g of the group $\mathcal{G}$ of symmetry transformations on M commute with $C_H(t)$

$$C_H(t) = gC_H(t)g^{-1} \qquad \forall g \in \mathcal{G}. \tag{2.1}$$

In particular, if $\Gamma \in M$ is a point on a periodic orbit with period T, then its image $g\Gamma$ obtained by application of the transformation $g \in \mathcal{G}$ also lies on a periodic orbit with the same period. Indeed, transformation $C_H(T)$ maps every point Γ on the first orbit onto itself, and relation (2.1) implies that it does the same for any of the transformed points $g\Gamma$. We note that the orbit containing point $g\Gamma$ may or may not coincide with the orbit containing point Γ. We refer to the set of orbits reached by application of any group element to a fixed point on some specified but arbitrary primary orbit as a set of dynamically equivalent *orbits*.

What are the implications of the group symmetry for the set of dynamically equivalent periodic orbits? We distinguish two alternatives. First, no two points $g\Gamma, \forall g \in \mathcal{G}$ lie on the same orbit. Then, the number of elements in the set of dynamically equivalent orbits is equal to the number $|\mathcal{G}|$ of group elements. Second, some elements of $\mathcal{G}$ transform a point on the primary orbit into another point on the same orbit. This is the more interesting case, and we can apply a result obtained previously by one of the authors for the case of permutations [4]. Note that here, as in [4], we assume that $\mathcal{G}$ acts non-trivially on all periodic orbits. This is not necessarily always true and we will discuss the implications of the trivial action of $\mathcal{G}$ in the appendix.

We consider the set $\mathcal{S}$ of points on the primary orbit which are obtained from a given point Γ by symmetry transformations. The points in $\mathcal{S}$ are spaced equidistantly in time on the primary orbit. In other words, there exists a minimal time step T' such that $C_H(T')$ transforms any point in the set into the next (defined by following the trajectory in the direction of increasing time), and there exists a group element which does the same. Hence, all points in $\mathcal{S}$ can be reached from any given point in $\mathcal{S}$ by iterating a symmetry transformation $g_c \in \mathcal{G}$ which connects neighbouring points. We note that the group element g_c is equivalent to $C_H(T')$ only on the primary orbit and not on all points in M.

To prove this statement, we order the points in $\mathcal{S}$ in the manner in which they are reached as the primary orbit is traversed in the direction of increasing time. Of all the nearest neighbours in $\mathcal{S}$ defined by this ordering, let Γ_0 and Γ_1 be two points which are separated by the shortest time interval T'. (The choice of the pair may not be unique.) There then exists a map $C_H(T')$ which takes Γ_0 into Γ_1. By definition of set $\mathcal{S}$, there also exists a group element g_c such that $\Gamma_1 = g_c\Gamma_0$. However, for any integer κ, the point $C_H(T')^\kappa\Gamma_0$ also lies on the primary orbit. By repeated application of equation (2.1), it follows that this point can equivalently be written as $g_c^\kappa\Gamma_0$. Thus, all powers of g_c lead to points on the primary orbit, i.e. points in $\mathcal{S}$. However, the points $g_c^\kappa\Gamma_0$ with integer κ must also be *all* the points in $\mathcal{S}$. Indeed, if there were another point Γ' in $\mathcal{S}$ which cannot be written as $g_c^\kappa\Gamma_0$ with integer κ, it must lie on the primary orbit between a pair of points (Γ_m, Γ_{m+1}, say) which do have this form. However, then the time needed to reach Γ' from Γ_m would be shorter than T', in contradiction to the assumption.

Let k be the smallest integer for which g_c^k equals the unit element of the group. Then, $kT' = T$. Moreover, we have found that a cyclic subgroup $C_k^c \in \mathcal{G}$ of order k generated by g_c defines the group symmetry on the primary orbit. We note that because of equation (2.1), this statement is independent of the choice of the starting point on the primary orbit.

We now use simple group properties to extend this result to all orbits in the set of dynamically equivalent orbits. Let $\mathcal{H}_c \subseteq \mathcal{G}$ be the subgroup of elements of $\mathcal{G}$ which commute with g_c. We note that $C_k^c \subseteq \mathcal{H}_c$. The set $\{g g_c g^{-1}, \forall g \in \mathcal{G}\}$ forms a conjugate class of the group. Each element is covered $|\mathcal{H}_c|$ times as g runs through the group. Each element raised to the power k equals the unit element of the group.

We immediately conclude the following.

(i) With g_c as the elementary group operation on the primary orbit, there exist $l = |\mathcal{H}_c|/k$ dynamically equivalent orbits with the same elementary group operation. Indeed, applying any element g_h in $\mathcal{H}_c$, which cannot be written as a power of g_c, to a point on the primary orbit leads to an orbit which differs from the primary orbit. On this orbit, $g_h g_c g_h^{-1}$ has the same function that g_c has on the primary orbit and $g_h g_c g_h^{-1} = g_c$ by definition of $\mathcal{H}_c$.

(ii) Each element of the conjugate class of g_c is, by the same reasoning, the elementary group operation on another subset of l dynamically equivalent orbits.

This makes a total of $|\mathcal{G}|/k$ distinct orbits which comprise the set of dynamically equivalent orbits generated from the primary orbit. We note that the primary orbit does not play a distinct role in the set; it was introduced only in order to generate the set. Any orbit in the set would have served the same purpose. This shows that we have established an equivalence relation between orbits belonging to the same set. This equivalence relation is generated by the conjugate class generated by g_c. The set of dynamically equivalent orbits consists of subsets each containing l orbits with the *same* elementary symmetry operation. For the application of our result in the next section, this last point is irrelevant because the trace operation eliminates the distinction between different members of the conjugate class.

We note that the central element of our theory is given by a piece of the primary orbit which connects a point Γ_0 to the neighbouring point $g_c\Gamma_0$. We refer to this as the elementary trajectory. Application of group operations to the elementary trajectory generates other trajectories; the sum of which equals the set of dynamically equivalent periodic orbits. The symmetry group thus establishes an equivalence relation between trajectories which are pieces of periodic orbits belonging to the same set. It follows that these pieces all have the same topological properties and, in particular, the same Maslov index. They also have the same dynamical properties such as the action along the trajectory or the monodromy matrix. All this follows directly from equation (2.1).

It also follows that for a non-cyclic symmetry group, the sets of dynamically equivalent periodic orbits must always have more than one member. Even for cyclic groups, such sets will, in general, occur.

Having found a classification scheme for a set of dynamically equivalent orbits, we now introduce a more general concept. The set of algebraically equivalent orbits comprises all sets of dynamically equivalent orbits which are characterized by the same conjugate class within $\mathcal{G}$, i.e. the elementary group operations are from the same class on all these orbits. Generically, this is an infinite set while all finite sets of dynamically equivalent orbits comprising it have the same group-theoretical structure, i.e. the same groups $\mathcal{H}_c$, C_k^c and the same finite number of elements k. We make use of this concept in section 4 when we break the periodic-orbit sum into contributions from different algebraically equivalent sets and, within each set, from different dynamically equivalent sets.

3. The trace formula

In the formulation of the trace formula, we shall closely follow [4] and shall therefore limit ourselves to sketching the main steps while emphasizing the additional aspects that enter the problem due to symmetry adaptation.

Let $\rho(E)$ be the level density of the quantum Hamiltonian H. Then,

$$\rho(E) = \frac{(-)}{\pi} \operatorname{Im} \operatorname{trace} G(E) \tag{3.1}$$

where $G(E) = (E^+ - H)^{-1}$ is the advanced Green function associated with Hamiltonian H.

Owing to the invariance of the Hamiltonian under transformations $g \subset \mathcal{G}$, the quantum-mechanical problem will split into independent problems defined on the subspaces of Hilbert space associated with the various irreducible group representations [5]. In certain cases, particular IR will be of interest while in others all IR will be relevant and appear in the spectrum. For the purpose of the analysis of the spectra, we prefer to consider the different symmetries separately. In any case, this will imply that the trace is to be taken over states with the correct symmetry only.

For the semi-classical treatment, we make one additional assumption: $\mathcal{G}$ is a group of point transformations, i.e. a group of transformations that map the configuration space onto itself. (The point transformations need not but may be linear.) While this simplification is not essential to the classification of orbits presented in the previous section, it allows us to readily obtain periodic-orbit sums from a path-integral formulation, and it covers the most obvious cases of interest.

Using the general definition of projectors in terms of characters [5], we write for the projection operator

$$P^f = \frac{|f|}{|\mathcal{G}|} \sum_{g \subset \mathcal{G}} \chi^f(g) g^{-1} \tag{3.2}$$

where χ^f indicates the IR of $\mathcal{G}$ labelled by f. This projection operator can be rewritten as

$$P^f = \frac{|f|}{|\mathcal{G}|} \sum_i \frac{1}{|\mathcal{H}_i|} \sum_{g \in \mathcal{G}} \chi_i^f g^{-1} g_i^{-1} g. \tag{3.3}$$

Here, the first sum is taken over all (conjugate) classes and g_i is some representative of a given class. The character χ^f is a class function as indicated by replacing the dependence on the group element by the index i for the conjugate class.

We now have to introduce the projection operator into trace equation (3.1) and obtain

$$\rho^f(E) = \frac{(-)}{\pi} \operatorname{Im} \operatorname{trace}[G(E) P^f]. \tag{3.4}$$

Here, due to the cyclic property of the trace and the fact that the projection operator is idempotent, this operator appears only once, though formally it ought to appear on both sides in order to guarantee that projections on the appropriate invariant subspace are taken. For the same reason, we will now see that the sum over all group elements that appears in the projection operator drops out in the trace. If $\Gamma = (q, p)$ with $q = \{q_1, q_2 \ldots q_n\}$, and similarly for p, we may write explicitly

$$\begin{aligned}\rho^f(E) &= \frac{(-)}{\pi} \frac{|f|}{|\mathcal{G}|} \sum_i \frac{1}{|\mathcal{H}_i|} \chi_i^f \operatorname{Im} \sum_{g \in \mathcal{G}} \int \mathrm{d}q \, \mathrm{d}q' \, \langle q | G(E) | q' \rangle \delta(g_i g q, g q') \\ &= \frac{(-)|f|}{\pi} \sum_i \frac{1}{|\mathcal{H}_i|} \chi_i^f \operatorname{Im} \int \mathrm{d}q \, \mathrm{d}q' \, \langle q | G(E) | q' \rangle \delta(g_i q, q'). \end{aligned} \tag{3.5}$$

Here, as in [4], we use the configuration-space representation of the class generator g_i in terms of a product of Dirac δs in vector notation. Note that the reduction to a sum over conjugate classes is not simply related to the reduction we found to classify the periodic orbits in the previous section, despite its being the same decomposition.

Using Feynman path integrals and following [4] closely for the stationary-phase approximation, we can rewrite each term of the sum in equation (3.5) as a sum over all classical trajectories that lead at a given energy E from any point $\Gamma \in M$ to $g_i \Gamma$ and obtain

$$\begin{aligned}&\int \mathrm{d}q \, \mathrm{d}q' \, \langle q | G(E) | q' \rangle \delta(g_i q, q') \\ &\qquad = \sum_{\text{classical trajectories}} \int \frac{\mathrm{d}q}{\dot{q}} \frac{2\pi}{(2\mathrm{i}\pi\hbar)^{(n+1)/2}} F^{-1/2} \exp\left(\frac{\mathrm{i}S(E)}{\hbar} - \frac{\mathrm{i}\pi\mu}{2}\right). \end{aligned} \tag{3.6}$$

Each trajectory in sum (3.6) is determined by the energy E, the equations of motion and by the end points q and $g_i q$ in configuration space. Moreover, $S(E)$ is the action along the trajectory from Γ to $g_i \Gamma$, μ is the Maslov index (an integer) and F is the determinant of the monodromy matrix. This matrix has the form $1 - A$ where A describes the map of the infinitesimal area perpendicular to the trajectory at q onto the corresponding area at $g_i q$. As shown in section 2, each trajectory in equation (3.6) is part of a periodic orbit. The integral in equation (3.6) extends over all points along the projection of the periodic trajectory on configuration space.

4. The periodic-orbit sum

In this section, we proceed to give expression (3.5) in its entirety with the expansion obtained for the stationary-phase approximation for integral (3.6) in simple form in terms of the classical periodic orbits, i.e. we use the classification of section 2 to simplify the result as far as possible. To do this, we proceed in two steps. First, we analyse a single term in the sum on the right-hand side of equation (3.6) in order to clearly display the role of the group element g_i in such a term. Then, we combine the sum over classes in equation (3.5) with the sum over orbits in equation (3.6) to obtain the final result in its simplest form.

As shown in section 2, a trajectory connecting points Γ and $g_i\Gamma$ must be part of a periodic orbit. Hence, it must be possible to express S, μ and $F = 1 - A$ in terms of the corresponding quantities S_0, μ_0 and $F_0 = 1 - A_0$ corresponding to the elementary trajectory connecting Γ and $g_c\Gamma$, where g_c is the elementary group operation (see section 2) on the periodic orbit of which the trajectory is part. We first note that with S_p, μ_p and $F_p = 1 - A_p$ denoting the action, the Maslov index and the monodromy matrix for a single traversal of the entire orbit, we have $S_0 = S_p/k$, $\mu_0 = \mu_p/k$ and $A_0 = A_p^{-k}$. Then $S = \kappa S_0$, $\mu = \kappa\mu_0$ and $A = A_0^{\kappa}$ for a single traversal of the trajectory corresponding to the κ-fold application of g_c to a point on the periodic orbit for which g_c is the elementary group operation. These relations hold because the quantities S_0, μ_0, A_0, S, μ and A are independent of the initial point chosen on the orbit.

Allowing for ν traversals of the entire periodic orbit as we reach $g_i\Gamma$ from Γ, we find that the contribution of this trajectory will read as

$$\frac{2\pi T}{(2\pi i\hbar)^{(n+1)/2}} \det{}^{-1/2}(1 - A_0^{\nu k+\kappa}) \exp[i(\nu k + \kappa)(S_0/\hbar - \pi\mu_0/2)]. \tag{4.1}$$

The integral in equation (3.6) would be obtained by summing over all ν to take care of the multiple traversals and over all periodic orbits determining the value of κ appropriately. However, we can obtain a simpler expression for the entire trace, as given in equation (3.5), if we take into account the following considerations.

The group element g_i in equation (3.6) does not determine the algebraic equivalence class to which the periodic orbits belong. This is readily understood if we take into account that not all elements of C_k^i belong to the same conjugate class. Actually, it is only the elementary group operation g_c on a given orbit which characterizes both the cyclic group and the algebraic equivalence of orbits. Thus $g_i = g_c^{\kappa}$ will certainly lead to a trajectory on all orbits for which g_c is the elementary group operation. Therefore, we would have to determine, for any given algebraic equivalence class, the value κ and the elementary group operation to which g_i belongs, if we wish to perform the sum over all i.

In practice, we invert the order of the summations. This implies that we have to consider a specific trajectory, characterized algebraically by the conjugate class c to which the corresponding periodic orbit belongs, and the power κ to which the elementary operation on this orbit is lifted, to reach the end point. This defines the conjugate class $j(c,\kappa)$ to which g_c^{κ} belongs. As this class is defined uniquely, we obtain a Kronecker delta $\delta(i, j(c,\kappa))$ in the final expression. Thus, we can supress the summation over i and replace the index of the character by $i(c,\kappa)$. As shown in the appendix, this no longer holds true for degenerate periodic orbits.

Thus, we can write equation (3.5) as

$$\rho(E) = \frac{(-)|f|}{\pi} \sum_c \frac{1}{k} \sum_{\kappa=1}^{k} \chi^f_{i(c,\kappa)} \sum_{d_c} \sum_{\nu=0}^{\infty} \mathrm{Im}\left[\frac{2\pi T}{(2\pi i\hbar)^{(n+1)/2}} \det{}^{-1/2}(1 - A_0^{\nu k+\kappa}) \right.$$
$$\left. \times \exp[i(\nu k + \kappa)(S_0/\hbar - \pi\mu_0/2)]\right]. \tag{4.2}$$

Note that not all dependencies on summation indices are explicitly displayed. Thus, for example, all dynamical quantities depend on d_c, and k depends on c.

We recall the structure of the summations in this final formula.

(i) The ν sum indicates the number of retracings

(ii) The d_c sum runs over an infinite set of dynamically different, but algebraically equivalent unstable periodic orbits. Note that exactly one of each set of dynamically equivalent orbits must appear; the representative can be chosen arbitrarily.

(iii) The κ sum identifies how far we follow a given orbit without retracings.

(iv) The c sum ranges over all conjugate classes.

The result obtained in this way has the advantage of clearly displaying the group-theoretical structure of the problem. The classification of periodic orbits achieved in section 2 was used to reduce the periodic-orbit sum to a sum over certain elementary time evolutions, thus greatly simplifying the situation. Note that the case where any group operation leads to a new periodic orbit is clearly included, but in this case the sum over κ vanishes as the retracings, beginning with one, take care of the entire situation. In keeping with our earlier statement, in a trace formula, there is no distinction between the action of identical and conjugate elementary operations.

5. Conclusions

We have investigated the influence of a discrete symmetry of a chaotic Hamiltonian on the structure of periodic orbits. We have found that the periodic orbits can be grouped into classes of algebraically equivalent orbits. Each equivalence class, in turn, is the union of classes of dynamically equivalent orbits. The orbits in each of the latter classes can be viewed as the union of a set of trajectories which, in turn, are connected by an equivalence relation generated by the symmetry group.

We have studied the consequences of these results for the semi-classical approximation to the density of states corresponding to a given irreducible representation of the symmetry group. Using the semi-classical approximation in a properly modified quantum-mechanical trace formula leads to a modification of the standard expression for the semi-classical periodic-orbit sum: the amplitudes with which individual orbits contribute to the sum depend on the irreducible representation chosen. We have used the above-mentioned classification of orbits to rewrite the periodic-orbit sum as a sum of contributions over classes. The resulting generic structure for the sum has one shortcoming: group theory alone does not allow us to say which classes actually occur in the sum with which frequency. This is determined by the dynamics of the system, and remains to be calculated individually for each system.

As stated in the introduction, we expect our approach to be useful for large groups. The most obvious case is that of the symmetric group. It arises in the treatment of bosonic or fermionic systems. Such systems may have internal degrees of freedom such as spin and/or isospin and the problem then is not quite as simple as the one treated in [4]. We can either construct more complicated projectors involving spin and then integrate the spin variables exactly, or we can replace the spin degrees of freedom by using more general orbital symmetries which are known to be isomorphic to the spin formulation [6]. (Similar procedures may be used in nuclear physics for fixed super multiplets [7, 8].) We prefer the latter course because the meaning of the semi-classical approximation inherent in the stationary-phase approximation is more obvious if only quantities that have a classical limit appear. The formalism developed in this paper has the required generality.

It will be interesting to extend these considerations to Hamiltonians which also depend on intrinsic degrees of freedom like spin or isospin and possess discrete symmetries only when *all* degrees of freedom are considered.

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Appendix

The degenerate case, i.e. the case where a symmetry operation leaves an entire periodic orbit p pointwise invariant may occur, although it is probably most important for groups with a small number of elements, for which the reduced phase-space method [9] works quite well. Nevertheless, we shall outline the necessary treatment for such a case. First, we note that the elements of $\mathcal{G}$ that leave a given periodic orbit pointwise invariant must form a subgroup $\mathcal{I} \subseteq \mathcal{G}$; all equivalent orbits will be invariant under an isomorphic group generated by conjugation with the element $g \in \mathcal{G}$ that transforms the original orbit on which $\mathcal{I}$ was defined into any of the equivalent orbits.

We now consider all elements $g \in \mathcal{G}$ which map the periodic orbit p onto itself (without necessarily leaving p pointwise invariant). These elements form a subgroup $\mathcal{G}_p$ of $\mathcal{G}$ which is no longer just the cyclic group generated by the elementary group operation on this orbit (which must still exist as the corresponding argument of section 2 continues to be valid once we exclude group elements that act trivially on p). Moreover, $\mathcal{I}$ is a normal subgroup of $\mathcal{G}_p$. This follows directly from the the fact that $\mathcal{I}$ is the largest subgroup of $\mathcal{G}_p$ which acts trivially on p, while p in turn may be considered an invariant space under the action of $\mathcal{G}_p$. The factor group $\mathcal{G}_p/\mathcal{I}$ must be isomorphic to the cyclic group generated by the elementary group operation on p. Note that this group need not be the same cyclic group that would occur for any of the generic orbits.

We now consider a left coset decomposition of $\mathcal{G}$ with respect to $\mathcal{G}_p$. The generators are not unique, but each of them will, when applied to the original orbit, take us to a different orbit, as all group elements that leave the orbit invariant have been factored out. Note that no new relations result from the symmetry of the orbit reached by the coset generator, as the invariance group of the new orbit is just the conjugate of the invariance group for the original orbit. From the above argument, we immediately conclude that the number of periodic orbits equivalent to p is $|\mathcal{G}_p| = k \times |\mathcal{I}|$, where k is the order of the cyclic group generated by the primitive group operation on p.

The upshot of these considerations is that the number of equivalent periodic orbits will be much smaller in the degenerate case than in the non-degenerate case. On the other hand, the sum in the projection operator still runs over all group elements. Thus, we will have to modify the final argument that combines the two sums in section 4. There we simply had to determine to which conjugate class a given power κ of the elementary group operation would belong. This was expressed by the index function $i(c, \kappa)$. Now this function can be multivalued and we instead need a multiplicity $m(i, c, \kappa)$ which counts how many elements of the set $g_c^\kappa \mathcal{I}$ lie in the conjugacy class i, and then formally perform the summation over i and c separately. For the non-degenerate case, this function would, for any c, κ combination, take the value 1 only once and otherwise would vanish. This fact allows us to eliminate one summation in section 4.

The very specific structure of the subgroups that occur limit the possible algebraic structures, and it is, in principle, again feasible to give an algebraic characterization of the equivalence classes for degenerate orbits. In this case, it seems rather likely that not all possibilities are realized and it is thus reasonable to work out the specific multiplicities for any case of interest.

In the present framework, there is no difference in the *form* of the stability factors for those periodic orbits which are left pointwise invariant by some $g \subset \mathcal{G}$, and for those other periodic orbits which do not have this property. This is because the calculation of the stability factors proceeds exactly in parallel in both cases. On the other hand, in the method of [9], pointwise-invariant periodic orbits require a special treatment because they lie on the boundary of the fundamental domain. The resulting stability factors are not obviously of the same form as found for periodic orbits in the interior of the fundamental domain, or as found in the present paper. Unfortunately, details of the calculation are not given in [9], and we have therefore not been able to trace the root of this difference.

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